

## ISOMETRY GROUP OF SASAKI-EINSTEIN METRIC

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Let  $(M, g)$  be a Sasaki-Einstein manifold of dimension  $2n+1$ ; equivalently its Kahler cone is a Kahler-Ricci flat cone. Let  $(X, J)$  be the underlying affine variety of its Kahler cone and denote  $\text{Aut}(X, J)$  to be its automorphism group; denote  $\text{Aut}_0(X, J)$  to be the identity component of  $\text{Aut}(X, J)$ . We prove the following result in this short paper,

**Theorem 1.** *The identity component of the holomorphic isometry group of  $(M, g)$  is the identity component of a maximal compact subgroup of  $\text{Aut}(X, J)$ .*

This answers a conjecture proposed in Martelli-Sparks-Yau [8] about the holomorphic isometry group of a Sasaki-Einstein metric; when a Sasaki-Einstein metric is *quasiregular*, this is proved in Martelli-Sparks-Yau ([8], Section 4.3). The statement itself can be viewed as a generalization of Mastushima's theorem [9] on a Kahler-Einstein metric on a Fano manifold, which asserts that the identity component of the isometry group of a Kahler-Einstein metric on a Fano manifold is the identity component of a maximal compact subgroup of its automorphism group. Unlike Fano case, a killing vector field of a Sasaki-Einstein metric does not have to be holomorphic; hence we can only assert the conclusion about holomorphic isometry group. A typical example is the odd dimensional  $(2n+1)$  round sphere whose identity component of isometry group is  $SO(2n+2)$ , but the holomorphic isometry group is  $U(n+1)$ . By a general result on Sasaki manifolds (see Theorem 8.18, Corollary 8.19 in [1]), a Killing vector field of a Sasaki-Einstein metric is (real) holomorphic unless on a round sphere or a 3-Sasaki structure (its Kahler cone is a hyper-Kahler cone and this is the counterpart of hyper-Kahler structure; it is always quasi-regular). Hence except these two special cases, the holomorphic condition in Theorem 1 can be dropped.

In this note we shall prove Theorem 1 when the Sasaki metric  $(M, g)$  is *irregular*. Given a Sasaki metric  $(M, g)$ , its *Reeb vector field*  $\xi$  is a holomorphic Killing vector field of  $(X, J, \bar{g})$ , where  $\bar{g}$  is the Kahler cone metric. We fix a maximal torus  $\mathbb{T}^k \subset \text{Aut}_0(X, J)$  such that its Lie algebra  $\mathfrak{t}$  contains  $\xi$ ; we can assume that the dimension  $k$  of  $\mathbb{T}^k$  is at least two without loss of generality (this is the case when  $\xi$  is irregular for example). Let  $K$  be a maximal compact subgroup of  $\text{Aut}(X, J)$  containing  $\mathbb{T}^k$  and we denote its Lie algebra as  $\mathfrak{h} = \text{Lie}(K)$ . The starting point is that the Reeb vector field is in the center of  $\mathfrak{h}$ , as in quasi-regular case [8].

**Proposition 0.1.** *The Reeb vector field  $\xi$  of Sasaki-Einstein metric  $(M, g)$  is in the center of  $\mathfrak{h} = \text{Lie}(K)$ .*

*Proof.* Let  $\mathfrak{z}$  be the center of  $\mathfrak{h}$ . And we can then write  $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}'$ . The Reeb vector fields form a convex subset of  $\mathfrak{t}$ , called *Reeb cone* and denoted by  $\mathcal{R}$ . As in [7], we shall mainly interested in the *normalized Reeb vector*

fields which lie in a hyperplane  $\mathcal{H}$  in  $\mathfrak{t}$  and we denote it as  $\mathcal{R}' = \mathcal{R} \cap \mathcal{H}$ . In [8] (see [5] for expository), it was proved that the volume functional  $V : \mathcal{R}' \rightarrow \mathbb{R}$  of a Sasaki structure depending only on the Reeb vector fields, and it is a convex functional in  $\mathcal{R}'$ ; moreover the Reeb vector field  $\xi$  of a Sasaki-Einstein metric has to be the (unique) critical point of the volume functional. Actually it was proved further that the volume functional is actually proper in  $\mathcal{R}'$  and hence such a minimizer always exists [7]. Clearly we can restrict our discussion on  $\mathfrak{z}$  and there is a unique minimizer, denoted as  $\xi_*$  of the volume functional when it is restricted to the normalized Reeb cone contained in  $\mathfrak{z}$ . It remains to show that  $\xi = \xi_*$ . When  $\xi_*$  is quasi-regular, this is proved in [8]. Hence we assume  $\xi_*$  is irregular and hence  $\dim \mathfrak{z} \geq 2$ . We can choose a sequence normalized Reeb vector fields  $\{\xi_n\}$  in  $\mathfrak{z}$  such that  $\xi_n \rightarrow \xi$  by a result of Rukimbira; moreover each  $\xi_n$  can be taken as quasi-regular (see [10] or Theorem 7.1.10 [1]). Now for any  $\zeta \in \mathfrak{t}'$ , we suppose  $\zeta$  satisfies the normalized condition such that for any normalized Reeb vector field  $\tilde{\xi}$ ,  $\tilde{\xi} + t\zeta$  is still a normalized Reeb vector field for (small) real number  $t$ . We then consider the volume functional  $v(t) = V(\xi_n + t\zeta)$ . We claim that  $V(\xi_n) \leq V(\xi_n + t\zeta)$  for small  $t$ . Clearly  $v(t)$  is a convex function of  $t$  and we only need to show that  $v'(0) = 0$ . Since  $\xi_n$  is quasi-regular and we can consider the quotient orbifold  $Z = M/\mathcal{F}_{\xi_n}$ . Then  $\mathfrak{t}'$  descends to a Lie subalgebra of  $\text{aut}_{\mathbb{R}}(Z)$ . Recall now the variation of the volume functional  $dV$  coincides with the Futaki invariant (up to a multiplication of a constant). Now recall that the Futaki invariant  $F_{\mathbb{C}} : \text{aut}(Z) \rightarrow \mathbb{C}$  is only nontrivial on the center of  $\text{aut}(Z)$  and in particular it vanishes on the complexification of  $\mathfrak{t}'$ . Hence it follows that  $dV_{\xi_n}(\zeta) = v'(0) = 0$  and the claim  $V(\xi_n) \leq V(\xi_n + t\zeta)$  is proved. By the smoothness of volume functional on Reeb vector fields, we know that  $V(\xi_*) \leq V(\xi_* + t\zeta)$  for any normalized  $\zeta \in \mathfrak{t}'$  and small  $t$ . It follows that  $dV_{\xi_*}(\zeta) = 0$  for any  $\zeta \in \mathfrak{t}'$ . It follows that  $\xi_*$  is also a critical point of  $V$  in  $\mathcal{R}'$  (hence minimizer of  $V$ ). By the uniqueness of minimizer in  $\mathcal{R}'$ ,  $\xi_* = \xi$ .  $\square$

Now we suppose  $\xi \in \mathfrak{z}$  and  $\dim(\mathfrak{z}) \geq 2$ . Let  $G$  be the identity component of the isometric group of  $(M, g)$  with Lie algebra  $\mathfrak{g}$ ; clearly  $\xi$  is also in the center of  $\mathfrak{g}$ . Now we can choose a sequence of normalized Reeb vector fields  $\xi_n$  which are quasi-regular and lie in  $\mathfrak{z}$  and the center of  $\mathfrak{g}$ . When  $n$  is sufficiently large, then we have the following,

**Proposition 0.2.** *For  $\xi_n$ , there exists a Sasaki-Ricci soliton  $g_n$  such that its underlying Kahler cone is  $(X, J)$  and its identity component of the isometric group is still  $G$ .*

*Proof.* This is really just the local deformation of Sasaki-Ricci solitons with Kahler cone fixed while with Reeb vector fields varied. The existence of such Sasaki-Ricci solitons follows from an argument of implicit function theory (in an  $G$ -invariant way). The argument of Theorem 4.1 ([7]) proves such a local deformation theory in an  $\mathbb{T}$ -invariant way; since  $\xi$  and  $\xi_n$  are all in the center of  $\mathfrak{g}$ , the same argument of Theorem 4.1 still applies with the maximal torus replaced by  $G$ . In particular, the isometry group of  $(M, g_n)$  contains  $G$ . Now by a general theorem of Grove-Kratcher-Ruh [6], we know that when  $n$  large enough, there is an inclusion, up to conjugation, of isometry

group of  $(M, g_n)$  into the isometry group  $G$  of  $(M, g)$  (see Lemma 8.2 [7] for example). It follows that the isometry group of  $(M, g_n)$  also has identity component  $G$ , up to conjugation.  $\square$

Hence we only need to prove that the identity component of isometry group  $G$  of  $(M, g_n, \xi_n)$  is the identity component of a maximal compact subgroup of  $\text{Aut}(X, J)$ , for sufficiently large  $n$ . This is a Calabi type theorem [4] and it proved by Tian-Zhu [11] for Kahler-Ricci solitons on Fano manifolds.

**Theorem 2** (Tian-Zhu). *Suppose  $(M, g, J)$  is Kahler-Ricci soliton on a Fano manifold  $(M, J)$ . Then the identity component of the isometry group of  $(M, g)$  is a maximal compact group of the identity component of  $\text{Aut}(M, J)$ .*

By a direct adaption of Tian-Zhu's argument, we have

**Proposition 0.3.** *For quasi-regular Sasaki-Ricci solitons  $(M, g_n, \xi_n)$ , the identity component of its isometry group is the identity component of a maximal compact subgroup of  $\text{Aut}(X, J)$ .*

*Proof.* Let  $K$  be a maximal group in  $\text{Aut}(X, J)$  such that  $\xi_n$  is in its Lie algebra  $\mathfrak{h}$  and let  $K_0$  be its identity component. Then by Proposition 0.1  $\xi_n$  is in  $\mathfrak{z}$ , the center of  $\mathfrak{h}$ . Since  $\xi_n$  is quasi-regular, it generates a  $U(1)$  action of  $(X, J)$  contained in  $K_0$ . Let  $Z = M/\mathcal{F}_{\xi_n}$  be the quotient orbifold and let the corresponding Kahler-Ricci soliton be  $h$ . The compact group  $K_0$ , modulo  $U(1)$  generated by  $\xi_n$ , then descends to a compact subgroup of the complex automorphism group  $\text{Aut}_0(Z)$ . By Tian-Zhu's theorem and its proof applied to  $(Z, h)$ , we know that  $K_0$  acts isometrically on  $(Z, h)$ . It then follows that  $K_0$  acts isometrically on  $(M, g_n, \xi_n)$ . Hence  $K_0$  coincides with  $G$ , the identity component of isometry group of  $(M, g_n, \xi_n)$ .  $\square$

Theorem 1 is then a corollary of Proposition 0.2 and Proposition 0.3.

Matsushima's theorem is on Lie algebra level and does not apply directly to a finite discrete subgroup which is not contained in the identity component. Bando-Mabuchi [2] proved that a Kahler-Einstein metric on a Fano manifold is unique modulo automorphisms; in particular, Kahler-Einstein metric must be invariant under a discrete subgroup  $\Gamma$  which is not in the identity component. The following short argument uses the same idea as in [2], but relies on the convexity of Ding's  $\mathcal{F}$ -functional, established by Berndtsson [3]; such an argument can also be applied directly to a Kahler-Ricci soliton.

**Proposition 0.4.** *Let  $(M, g)$  be a Kahler-Einstein metric (or a Kahler-Ricci soliton) on a Fano manifold  $(M, J)$ . Suppose  $\Gamma$  is a discrete subgroup in  $\text{Aut}(M, J)$  such that  $\Gamma \cap \text{Aut}_0(M, J) = \text{id}$ . Then  $g$  is  $\Gamma$ -invariant.*

*Proof.* We assume  $(M, g)$  is Kahler-Einstein for simplicity. The argument for Kahler-Ricci soliton is almost identical. Suppose  $\lambda \in \Gamma$  and consider  $\lambda^*g$ , which is a Kahler-Einstein metric on  $(M, J)$ . Note that  $\Gamma \subset \text{Aut}(M, J)$  and the Kahler class of  $g$  and  $\lambda^*g$  are both in  $c_1(M, J)$ , under appropriate normalization. Suppose  $g \neq \lambda^*g$ . Recall that in the space of Kahler potentials  $\mathcal{H}$ , there exists a unique geodesic  $\gamma(t)$  connecting  $g, \lambda^*g$  by a fundamental result of Chen. Recall that a Kahler-Einstein metric in  $c_1(M, J)$  is minimum of Ding's  $\mathcal{F}$ -functional, which is convex along geodesics in  $\mathcal{H}$ .

It follows that  $\mathcal{F}$ -functional is linear (constant) along  $\gamma(t)$ . By Berndtsson's theorem [3],  $\gamma(t)$  is generated by a holomorphic vector field  $\zeta$ . In particular, there exists a one-parameter subgroup  $\sigma_0$  generated by  $\zeta$  such that  $\sigma_0 = id, \sigma_1 = \lambda$ . This contradicts that  $\Gamma \cap Aut_0(M, J) = id$ . Similar argument applies to a Kahler-Ricci soliton  $(M, g)$  with  $\mathcal{F}$ -functional replaced by modified  $\mathcal{F}$ -functional, introduced by Tian-Zhu [12].  $\square$

One may wonder whether the above Bando-Mabuchi's result for Kahler-Einstein metrics on Fano manifolds holds or not for a Sasaki-Einstein metric. We believe this might not be the case in Sasaki setting due to the possible complexity of  $Aut(X, J)$ . The main point is that in Kahler setting, under the action of automorphism group (or discrete subgroup), the first Chern class (hence the Kahler class of Kahler-Einstein metric, modulo scaling) is invariant. In Sasaki setting, the Reeb vector field is also unique given a fixed Reeb cone; but we are not sure that such a Reeb cone is unique or not even within the Lie algebra  $\mathfrak{t}$  of a fixed (maximal) torus  $\mathbb{T} \subset Aut(X, J)$  (see Remark 2.9 in [7]). We ask the following problem,

**Question 3.** *Let  $(M, g)$  be a Sasaki-Einstein metric with a Reeb vector field  $\xi$ . Let  $K$  be a maximal compact subgroup of  $Aut(X, J)$  such that  $\mathfrak{h}$ , the Lie algebra of  $K$  contains  $\xi$  in its center. Let  $\Gamma$  be a discrete subgroup of  $K$  such that  $\Gamma \cap K_0 = id$ . Prove or disprove that  $(M, g)$  is  $\Gamma$ -invariant.*

We are not sure that whether the one-parameter group generated by  $\xi$  is in the center of  $K$  or not (we know  $\xi$  is in the center of  $\mathfrak{h}$ , but the proof does not carry to a finite discrete subgroup of  $K$ ). For any  $\lambda \in \Gamma$ , it induces an adjoint action  $Adj_\lambda : \mathfrak{h} \rightarrow \mathfrak{h}$ . If Reeb cone contained in  $\mathfrak{h}$  is unique, then by the uniqueness of Reeb vector field of a Sasaki-Einstein metric,  $Adj_\lambda(\xi) = \xi$ . It then follows that the one-parameter group generated by  $\xi$  is in the center of  $K$ . One can proceed to argue that  $(M, g)$  is  $\lambda$ -invariant as in Fano case. However, it could happen that  $\mathfrak{h}$  contains finite many (disconnected) Reeb cones corresponding exactly to a finite group  $\Gamma$ , and the adjoint action  $Adj_\lambda : \mathfrak{h} \rightarrow \mathfrak{h}$  permutes these Reeb cones. It would be an interesting question to understand whether this phenomenon can actually happen or not.

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#### REFERENCES

- [1] C. Boyer, K. Galicki, *Sasakian geometry*.
- [2] S. Bando, T. Mabuchi, *Uniqueness of Einstein Kahler metrics modulo connected group actions*. Algebraic geometry, Sendai, 1985, 11-40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.
- [3] B. Berndtsson, *A Brunn-Minkowski type inequality for Fano manifolds and the Bando-Mabuchi uniqueness theorem*, arxiv.org/abs/1103.0923.
- [4] E. Calabi, *Extremal Kähler metrics. II*. Differential geometry and complex analysis, 95-114, Springer, Berlin, 1985.
- [5] A. Futaki, H. Ono, G-F. Wang. *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds*, J. Diff. Geom. 83 (2009), 585-636.
- [6] K. Grove, H. Karcher, E. Ruh, *Group actions and curvature*. Invent. Math. 23 (1974), 31-48.
- [7] W-Y. He, S. Sun, *Frankel conjecture and Sasaki geometry*, arxiv.org/abs/1202.2589.

- [8] D. Martelli, J. Sparks, S-T. Yau. *Sasaki-Einstein Manifolds and Volume Minimisation*, Commun. Math. Phys. 280 (2007), 611-673.
- [9] Y. Matsushima, *Sur la structure du groupe d'homeomorphismes analytiques d'une certaine variete kehlérienne*. (French) Nagoya Math. J. 11 1957 145-150.
- [10] P. Rukimbira, *Chern-Hamiltons conjecture and K-contactness*, Houston J. Math. 21 (1995), no. 4, 709-718.
- [11] G. Tian, X-H. Zhu. *Uniqueness of Kähler-Ricci solitons*. Acta Math. 184 (2000), no. 2, 271-305.
- [12] G. Tian, X-H. Zhu. *A new holomorphic invariant and uniqueness of Kähler-Ricci solitons*. Comment. Math. Helv. 77 (2002), no. 2, 297-325.

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